

Factorization and entanglement in quantum systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2003 J. Phys. A: Math. Gen. 36 517

(<http://iopscience.iop.org/0305-4470/36/2/315>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.103

The article was downloaded on 02/06/2010 at 15:30

Please note that [terms and conditions apply](#).

Factorization and entanglement in quantum systems

Jon Eakins and George Jaroszkiewicz

School of Mathematical Sciences, University Park, University of Nottingham, Nottingham,
NG7 2RD, UK

Received 16 May 2002

Published 17 December 2002

Online at stacks.iop.org/JPhysA/36/517

Abstract

We discuss the question of entanglement versus separability of pure quantum states in direct product Hilbert spaces, and the relevance of the issue to physics. Different types of separability may be possible, depending on the particular factorization of the Hilbert space. A given orthonormal basis set for a Hilbert space is defined to be of type (p, q) if p elements of the basis are entangled and q are separable, relative to a given bi-partite factorization of that space. We conjecture that not all basis types exist for a given Hilbert space.

PACS numbers: 03.65.Ud, 98.80.Qc

1. Introduction

The phenomenon of entanglement is of central importance in the interpretation of quantum mechanics. Historically, entanglement was the focus of the famous Einstein–Podolsky–Rosen (EPR) paper [1], which suggested that standard quantum mechanics is an incomplete theory of physical reality. The central argument of the EPR paper was that more information about incompatible variables such as momentum and position could in principle be deduced about an entangled two-particle quantum state than quantum mechanics permits, effectively giving information about each particle separately, and therefore supporting a classical perspective.

The resolution of this ‘paradox’ is the observation that information extraction in quantum mechanics always comes at a cost: it is not possible to actually extract information about incompatible variables from a given state without destroying the state being looked at before the information extraction process is completed, and this invalidates the argument used by EPR [2].

An apparently unrelated issue is the following. Throughout the history of quantum mechanics, a constant topic of debate has been where the boundary between the classical and quantum worlds should be. We believe that there is now sufficient evidence to support the notion that there is no such boundary, and that the classical world view is no more than an emergent, i.e. effective, view of a universe which is entirely quantum mechanical in origin [3]. The evidence we cite is the near universal validity of the quantized-field approach to elementary particles, numerous experimentally observed violations of Bell inequalities and

galactic lensing. In the latter process, we can imagine observing (say) one photon per day over several years to build up patterns analogous to the interference bands seen in double slit experiments, the difference being that the scale of the process is cosmological rather than local. In addition to these more exotic applications, the validity of quantum principles is supported by the overwhelming success of quantum mechanics in applied physics, biology and chemistry, on both terrestrial and astrophysical scales.

2. Separability and factorization

With a recognition that the semi-classical observers of standard quantum mechanics should ideally be regarded as quantum systems themselves, it has become more fashionable to extend the quantum description to include them with the systems under observation. This can be done whilst maintaining a semi-classical perspective by writing a quantum state vector Ψ for an *OS* (combined observer plus system under observation) as a direct product $\Psi \equiv \theta \otimes \phi$, where θ represents a state of the observer *O* and ϕ represents a state of the system *S* under observation. Such a state will be called *separable*. In general, we shall use the word *separable* when we talk about states constructed from direct products of vectors, and *factorizable* when we refer to Hilbert spaces constructed from direct (tensor) products of (factor) Hilbert spaces. When applied to Hilbert spaces, the term *separable* refers to the possibility of finding a countable basis for it, regardless of any issue of factorizability.

When the dimensions of realistic Hilbert spaces which might model the universe are considered, then the *a priori* probability that a state chosen at random in such a space be separable is zero, as will be seen from our discussion of concurrency below. The observed separability of the universe into vast numbers of identifiably distinct subsystems is surprising from this point of view. However, this does not take into account the crucial role of dynamics, which imposes very specific constraints on which states are physically accessible in the course of time. For example, suppose all the possible outcomes of some quantum process are separable states. Then there will be zero probability of getting an entangled state outcome in that process.

In this article, classicity (or classicality) is regarded as synonymous with the possibility of making distinctions between different objects, such as different spatial positions, or physical subsystems. In quantum mechanics, entanglement may be regarded as a breakdown of such a possibility. When physicists discuss isolated systems within a wider universe, they invariably model the totality by separable states, with some of the factors representing states of the isolated systems and other factors representing the rest of the universe. The conventional procedure is then to ignore these other factors (the environment), and discuss only those factors representing the isolated systems. Certainly, it seems impossible to discuss experiments in physics without assuming that the states of interest are factored out from the rest of the environment. The development of decoherence theory has not altered this in the least. Separability is therefore as fundamental to quantum physics as entanglement.

This leads to the following question: given a finite-dimensional Hilbert space \mathcal{H} of dimension $d \equiv \dim \mathcal{H}$, when is it possible to think of a state Ψ in \mathcal{H} as a separable state? By this we mean we would like to know the circumstances which guarantee that Ψ is a tensor product of the form $\Psi = \psi \otimes \phi$, where ψ is some vector in some factor space \mathcal{H}_1 of \mathcal{H} and ϕ is another vector in another factor \mathcal{H}_2 of \mathcal{H} .

Let \mathcal{H} be a finite-dimensional Hilbert space of dimension $\dim \mathcal{H}$. If \mathcal{H} can be expressed in the bi-partite form

$$\mathcal{H} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)} \quad (1)$$

where $\mathcal{H}_i^{(d_i)}$, $i = 1, 2$ is a Hilbert space with dimension d_i , then we shall say that \mathcal{H} is *factorizable*. Clearly $\dim \mathcal{H}$ must itself be factorizable and given by the rule $\dim \mathcal{H} = d_1 d_2$.

3. Cosmological implications

This elementary result may have important cosmological implications. According to a number of authors [3–6] the universe is described by a time-dependent pure quantum state Ψ , an element in a Hilbert space \mathcal{H}_U of enormous but finite dimension. We note that the notion that the universe is a quantum system has been criticized principally on the grounds that there is no evidence that all physical systems must possess quantum states [7], and also because it appears inconsistent to discuss probabilities when there is only one universe. These arguments can be met with three counter-arguments: first, the absence of any boundary between the quantum and classical worlds and the empirical validity of quantum mechanics actually strongly support the notion that *all* systems must run on quantum principles, and so by extension does the universe; second, a pure state formalism eliminates the need for a density matrix approach to quantum cosmology; third, quantum probabilities make sense if they are interpreted correctly in terms of *predictions* about the possible future state of the universe made by physicists who are themselves part of the quantum universe. This is not inconsistent with the notion that the universe is in a definite state at present.

Given this quantum perspective about the universe, the apparently overwhelmingly classical appearance of the universe, with a classical looking spatial structure which permits the separation of vast numbers of subsystems of the universe spatially, is interpreted by us as evidence that the current state of the universe Ψ has separated into a vast number of factors. If this is true then \mathcal{H}_U must be factorizable into a vast number of factor spaces, and, therefore, $\dim \mathcal{H}_U$ itself must be highly factorizable. In particular, the Hilbert space of the universe cannot have prime dimension according to this scenario.

4. Factorizable Hilbert spaces

Hilbert spaces with a high degree of factorizability are readily constructed. Recent approaches to fundamental physics inspired by spin networks and quantum computation [5, 8] consider \mathcal{H}_U to be the direct product of a (usually vast) number N of qubit Hilbert spaces, namely

$$\mathcal{H}_U = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)} \otimes \dots \otimes \mathcal{H}_N^{(2)} \quad (2)$$

and then $\dim \mathcal{H}_U = 2^N$. None of the individual qubit factor spaces $\mathcal{H}_i^{(2)}$ is factorizable, so that (2) represents a complete, or maximal, factorization of \mathcal{H}_U . We shall call such a qubit factorization a *primordial factorization*. In the most general case, a primordial factorization will be of the form

$$\mathcal{H} = \mathcal{H}_1^{(p_1)} \otimes \mathcal{H}_2^{(p_2)} \otimes \dots \otimes \mathcal{H}_N^{(p_N)} \quad (3)$$

where the p_i are prime numbers and $\dim \mathcal{H} = p_1 p_2 \dots p_N$. If the *factorizability* ζ of \mathcal{H} is defined as the ratio $N / \dim \mathcal{H}$ then qubits provide the maximum factorizability for a given N , i.e. $\zeta = N/2^N$. Qubits are favoured by various authors because they represent the most elementary attributes of logic, that is, ‘yes’ and ‘no’ (or equivalently, ‘true’ and ‘false’) can be identified with the two elements of a qubit ‘spin-up’, ‘spin-down’ basis.

Given an N -qubit system with $N > 2$ then it is possible to consider *partial factorizations* with *splits* of \mathcal{H} of the form

$$\mathcal{H} = \mathcal{H}^{(2^n)} \otimes \mathcal{H}^{(2^{N-n})} \quad (4)$$

where

$$\begin{aligned}\mathcal{H}^{(2^n)} &\equiv \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)} \otimes \dots \otimes \mathcal{H}_n^{(2)} \\ \mathcal{H}^{(2^{N-n})} &\equiv \mathcal{H}_{n+1}^{(2)} \otimes \mathcal{H}_{n+2}^{(2)} \otimes \dots \otimes \mathcal{H}_N^{(2)}\end{aligned}\quad (5)$$

and variants of this theme. If a given partial factorization has two factors, such as in (4) we shall call this a *bi-partite factorization*.

5. Separability of states

We now discuss a necessary and sufficient condition for the separability of a state relative to a given bi-partite factorization.

Let $\mathcal{B}_a^{(d_a)} \equiv \{|i\rangle_a : i = 1, \dots, d_a\}$ be an orthonormal basis for factor space $\mathcal{H}_a^{(d_a)}$. Orthonormality is not necessary for our theorem below to hold but is useful in subsequent discussions. Given that \mathcal{H} is factorizable in the form (1) then an orthonormal basis for \mathcal{H} is $\mathcal{B} \equiv \{|i\rangle_1 \otimes |j\rangle_2 : i = 1, 2, \dots, d_1, j = 1, 2, \dots, d_2\}$. Such a basis will be called a *factorizable basis*. Any state $|\Psi\rangle$ in \mathcal{H} can then be written in the form

$$|\Psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} C_{ij} |i\rangle_1 \otimes |j\rangle_2 \quad (6)$$

where the coefficients C_{ij} are complex and form the components of a complex $d_1 \times d_2$ matrix called the *coefficient matrix*. It is relatively easy to prove the following theorem:

Theorem. *The state $|\Psi\rangle$ is separable relative to the factorizable basis \mathcal{B} if and only if the coefficient matrix satisfies the micro-singularity condition*

$$C_{ij}C_{ab} = C_{ib}C_{aj} \quad (7)$$

for all possible values of the indices.

A proof involving the concept of *concurrence* is given in [9]. For example, a state $|\Psi\rangle$ in a two-qubit system of the form

$$|\Psi\rangle = \alpha|1\rangle_1 \otimes |1\rangle_2 + \beta|1\rangle_1 \otimes |2\rangle_2 + \gamma|2\rangle_1 \otimes |1\rangle_2 + \delta|2\rangle_1 \otimes |2\rangle_2 \quad (8)$$

is separable if and only if $\alpha\delta = \beta\gamma$, which can be readily verified.

There are two points to make here. First, given a factorizable basis, a coefficient matrix chosen at random will almost certainly not be micro-singular, simply because for large dimensions, there will be a vast number of micro-singularity conditions (7) to satisfy. The number N_C of such conditions will in general be given by $N_C = \frac{1}{4}d_1(d_1 - 1)d_2(d_2 - 1) \sim \frac{1}{4}\dim^2\mathcal{H}$ for large $\dim\mathcal{H}$. This is why the existence of separability in a universe which is running on quantum principles should come as a surprise. Rather than envisage entanglement as an extraordinary phenomenon, we should perhaps ask why the degree of separability in the current epoch of the universe is so relatively large. We envisage that, given a fully quantum universe which was jumping from one quantum state to another, most of these states should be entangled, unless there is some very special reason for separability. A related issue is the idea, consistent with recent developments in quantum gravity, that space itself is an emergent attribute of a completely quantum universe [3, 10]. It is hard to understand how this attribute could emerge unless the states of the universe in the epoch were highly separable and remained so under the influence of extraordinary dynamical laws. Without separability, there can be no notion of classicity, and without any form of classicity, the concept of space itself cannot be formulated. Position in space is, after all, synonymous with the classical statement that *this*

object is *here* and not *there*. As the *EPR* discussion shows, such a statement is not possible for entangled states. Moreover, from this viewpoint, the expansion of the universe may be taken as some indicator that, far from being of very low probability, separability is actually increasing, suggesting that the current dynamics of the universe is somehow organizing a greater degree of *classicity* (or separability) with time.

To illuminate the scale of the problem of explaining the current separability of the universe, a simple estimate of the lowest realistic dimension d_U of the Hilbert space \mathcal{H}_U of the universe gives $d_U \gtrsim 2^{10^{180}}$, which is based on the supposition that each Planck volume in the visible universe contains one elementary qubit. More realistic estimates would certainly increase this estimate dramatically. The set of separable states in \mathcal{H}_u is a set of measure zero, with the number of concurrency conditions being proportional to d_U^2 for large dimensions. We should ask, therefore, why does the universe have so much apparent separability, that is, why can physicists investigate isolated systems at all?

The second point is that separability depends on the choice (if any) of the factorization of the total Hilbert space \mathcal{H} . Consider the Hilbert space $\mathcal{H} \equiv \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)} \otimes \mathcal{H}_3^{(2)}$ where each $\mathcal{H}_i^{(2)}$ represents a qubit space. Now rearrange \mathcal{H} in the bi-partite form

$$\mathcal{H} = \mathcal{H}_A^{(4)} \otimes \mathcal{H}_3^{(2)} \quad (9)$$

where $\mathcal{H}_A^{(4)} \equiv \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$, with factorizable basis

$$\mathcal{B}_1 \equiv \{|ij\rangle_A \otimes |k\rangle_3 : 1 \leq i, j, k \leq 2\}. \quad (10)$$

Then a separable state $|\Psi\rangle$ relative to this factorization and this basis will be of the form

$$|\Psi\rangle = (a|11\rangle_A + b|12\rangle_A + c|21\rangle_A + d|22\rangle_A) \otimes (\alpha|1\rangle_3 + \beta|2\rangle_3) \quad (11)$$

where the coefficients a, b, \dots, β are complex, giving the coefficient matrix

$$\begin{array}{cc} \otimes & \mathbf{1}_3 & \mathbf{2}_3 \\ \mathbf{11}_A & a\alpha & a\beta \\ \mathbf{12}_A & b\alpha & b\beta \\ \mathbf{21}_A & c\alpha & c\beta \\ \mathbf{22}_A & d\alpha & d\beta \end{array} \quad (12)$$

which clearly satisfies micro-singularity. In this matrix, the top row and the left-most column label basis vectors for the different factor spaces, and the other terms represent the coefficients of their tensor products, i.e. the actual elements of the coefficient matrix.

Now we may also write the Hilbert space in the alternative bi-partite form

$$\mathcal{H} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_B^{(4)} \quad (13)$$

where $\mathcal{H}_B^{(4)} \equiv \mathcal{H}_2^{(2)} \otimes \mathcal{H}_3^{(2)}$. We note for example

$$|12\rangle_A \otimes |1\rangle_3 = |1\rangle_1 \otimes |21\rangle_B \quad (14)$$

and so on. Then the above state is given by

$$|\Psi\rangle = a\alpha|1\rangle_1 \otimes |11\rangle_B + a\beta|1\rangle_1 \otimes |12\rangle_B + \dots \quad (15)$$

giving the alternative coefficient matrix

$$\begin{array}{cc} \otimes & \mathbf{11}_B & \mathbf{12}_B & \mathbf{21}_B & \mathbf{22}_B \\ \mathbf{1}_1 & a\alpha & a\beta & b\alpha & b\beta \\ \mathbf{2}_1 & c\alpha & c\beta & d\alpha & d\beta \end{array} \quad (16)$$

which clearly does not satisfy micro-singularity. Therefore, a state separable relative to one factorization of \mathcal{H} is not necessarily separable relative to another factorization of \mathcal{H} .

Mathematicians tell us that separable and entangled states can be transformed into each other via unitary transformations, whereas physicists regard their differences as physically significant in the right context.

Apart from having consequences for the question of separability on emergent scales, this may have another cosmological implication. Given that the separability of the state of the universe is meaningful only relative to a special factorization of \mathcal{H} , this suggests that there is a *preferred* basis for \mathcal{H} before each jump. If the universe can jump only into an eigenstate of some specific complete set of observables [3], then that preferred basis will be the set of possible eigenstates (i.e. the possible outcomes) of that complete set of observables. This complete set may change with each jump, but nevertheless this picture must still hold. There must be something extraordinarily special about the selection of this set of operators. There must be something extraordinarily special about the selection of this set of observables. In a fully quantized universe running as a quantum automaton, this choice cannot be made by any external agency. Given that the number of independent Hermitian operators is of the order $d_{\mathcal{H}}^2$ [11], then there must be some as yet unknown and very specific laws which determine the operators responsible for the separability of the universe in the current epoch.

6. Separations and entanglements

An important feature of quantum theory is that individual elements of a factorizable space are not in general separable relative to a primordial factorization. If ψ is an element of $\mathcal{H}_1^{(d_1)}$ and ϕ is an element of $\mathcal{H}_2^{(d_2)}$, then $\Psi \equiv \psi \otimes \phi$ is a separable element of $\mathcal{H} \equiv \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)}$. We shall say that Ψ is *separable relative to the* $(\mathcal{H}_1^{(d_1)}, \mathcal{H}_2^{(d_2)})$ *factorization of* \mathcal{H} . It is a particularly important fact that the separability of Ψ relative to $(\mathcal{H}_1^{(d_1)}, \mathcal{H}_2^{(d_2)})$ does not depend on the choice of basis for $\mathcal{H}_1^{(d_1)}$ or for $\mathcal{H}_2^{(d_2)}$, that is, this separation is invariant to (local) unitary transformations of basis for $\mathcal{H}_1^{(d_1)}$ and $\mathcal{H}_2^{(d_2)}$ separately. If an element Θ of \mathcal{H} is not *separable relative to a* $(\mathcal{H}_1^{(d_1)}, \mathcal{H}_2^{(d_2)})$ *factorization of* \mathcal{H} then we shall say that Θ is *entangled relative to the* $(\mathcal{H}_1^{(d_1)}, \mathcal{H}_2^{(d_2)})$ *factorization of* \mathcal{H} . This too is a basis independent concept.

Now consider an arbitrary orthonormal basis \mathcal{B} for \mathcal{H} . This will have $d \equiv d_1 d_2$ elements $\beta_i, i = 1, 2, \dots, d$. The question we ask now is, how many of these are separable relative to the factorization $(\mathcal{H}_1^{(d_1)}, \mathcal{H}_2^{(d_2)})$ of \mathcal{H} and how many are entangled? If q of the β_i are separable and $p = d - q$ are entangled, that is, not separable relative to $(\mathcal{H}_1^{(d_1)}, \mathcal{H}_2^{(d_2)})$, then we shall say that \mathcal{B} is of type (p, q) . If $q = d$ then \mathcal{B} is a *completely separable basis relative to the factorization* $(\mathcal{H}_1^{(d_1)}, \mathcal{H}_2^{(d_2)})$ and if $p = d$ then \mathcal{B} is a *completely entangled basis relative to this factorization*. Otherwise, \mathcal{B} is a *partially separable (or partially entangled) basis relative to the factorization* $(\mathcal{H}_1^{(d_1)}, \mathcal{H}_2^{(d_2)})$.

If in such a bi-partite factorization, each of the factor spaces $\mathcal{H}_1^{(d_1)}, \mathcal{H}_2^{(d_2)}$ could in principle be factorizable into two or more factors, then this would lead to a natural extension of this sort of classification of bases, which is left to the reader to explore.

We now discuss an example which suggests that not every type of partially factorizable basis exists. This example is a two-qubit system, so that $d = 4$, and \mathcal{H} has the primordial factorization

$$\mathcal{H} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)} \quad (17)$$

where the $\mathcal{H}_i^{(2)}, i = 1, 2$ are individual two-dimensional qubit Hilbert spaces. Without loss of generality, we shall work with a specific choice of basis for each factor Hilbert space.

For qubit i , $\{|0\rangle_i, |1\rangle_i : i = 1, 2\}$ is an orthonormal basis for $\mathcal{H}_i^{(2)}$. Then we define

$$|ij\rangle \equiv |i\rangle_1 \otimes |j\rangle_2 \quad 0 \leq i, j \leq 1. \quad (18)$$

We shall give examples of type (0, 4), (2, 2), (3, 1) and (4, 0) bases for \mathcal{H} , and then a proof that type (1, 3) does not exist.

Type (0, 4). With the above notation, a completely factorizable basis, that is, a type (0, 4) basis $\mathcal{B}_{0,4}$ is given by

$$\mathcal{B}_{0,4} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}. \quad (19)$$

Type (2, 2). With the same notation as above, a type (2, 2) basis for \mathcal{H} is given by

$$\mathcal{B}_{2,2} = \left\{ |00\rangle, |11\rangle, \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \right\}. \quad (20)$$

Type (3, 1). Relative to the basis $\mathcal{B}_{0,4}$ given above, a type (3, 1) orthonormal basis for \mathcal{H} is given by

$$\mathcal{B}_{3,1} = \left\{ |00\rangle, \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle, \frac{1}{\sqrt{2}}|11\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle, \frac{1}{\sqrt{2}}\{|01\rangle - |10\rangle\} \right\}. \quad (21)$$

Type (4, 0). Relative to the basis $\mathcal{B}_{0,4}$ given above, a type (4, 0) orthonormal basis for \mathcal{H} is given by

$$\mathcal{B}_{4,0} = \left\{ \frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}, \frac{1}{\sqrt{2}}\{|00\rangle - |11\rangle\}, \frac{1}{\sqrt{2}}\{|01\rangle + |10\rangle\}, \frac{1}{\sqrt{2}}\{|01\rangle - |10\rangle\} \right\}. \quad (22)$$

The existence of type (3, 1) partially factorizable bases makes it surprising that no type (1, 3) basis can exist. Although intuitively obvious, a proof is surprisingly long:

Theorem. *No type (1, 3) basis of a two-qubit Hilbert space exists relative to the primordial factorization.*

Proof. Let η_1, η_2 and η_3 be three mutually orthogonal vectors which are separable relative to the primordial factorization $\mathcal{H} \equiv \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$ of a two-qubit Hilbert space \mathcal{H} . By definition these vectors are of the form

$$\eta_i = \psi_i \otimes \phi_i \quad i = 1, 2, 3 \quad (23)$$

where $\psi_i \in \mathcal{H}_1^{(2)}$ and $\phi_i \in \mathcal{H}_2^{(2)}$. None of the factor vectors ψ_i, ϕ_i can be zero, since we require

$$(\eta_i, \eta_i) = (\psi_i, \psi_i)_1 (\phi_i, \phi_i)_2 > 0 \quad i = 1, 2, 3 \quad (24)$$

where subscripts on inner products refer to the corresponding factor space.

Mutual orthogonality gives the three conditions

$$\begin{aligned} (\psi_1, \psi_2)_1 (\phi_1, \phi_2)_2 &= 0 \\ (\psi_1, \psi_3)_1 (\phi_1, \phi_3)_2 &= 0 \\ (\psi_2, \psi_3)_1 (\phi_2, \phi_3)_2 &= 0. \end{aligned} \quad (25)$$

Now define $A_{ij} \equiv (\psi_i, \psi_j)_1$, $B_{ij} \equiv (\phi_i, \phi_j)_2$ for $1 \leq i < j \leq 3$. First, we show that not all three of the A_{ij} can be zero. Suppose this were true. Then

$$A_{12} = 0 \quad \Rightarrow \quad (\psi_1, \psi_2)_1 = 0. \quad (26)$$

Since none of the ψ_i can be zero and since $\mathcal{H}_1^{(2)}$ is two dimensional, we deduce that ψ_1 and ψ_2 form an orthogonal basis for $\mathcal{H}_1^{(2)}$. Hence we may write

$$\psi_3 = a\psi_1 + b\psi_2 \quad |a|^2 + |b|^2 > 0. \quad (27)$$

Then

$$\begin{aligned} A_{13} = 0 &\Rightarrow a = 0 \\ A_{23} = 0 &\Rightarrow b = 0 \end{aligned} \quad (28)$$

which contradicts (27). Likewise, not all the B_{ij} can be zero.

Without loss of generality, the only way to satisfy the mutual orthogonality conditions (25) is to have $A_{12} = A_{13} = B_{23} = 0$ and $A_{23} \neq 0$. (By symmetry, any other choice of the A_{ij}, B_{jk} would be just as good.)

With these conditions assumed, we use the condition $A_{12} = 0$ as before to deduce condition (27). The condition $A_{13} = 0$ gives $a = 0$ and so we deduce $b \neq 0$.

Similarly, $B_{23} = 0$ implies that ϕ_2 and ϕ_3 form an orthogonal basis for $\mathcal{H}_2^{(2)}$, and therefore we may write

$$\phi_1 = c\phi_2 + d\phi_3 \quad |c|^2 + |d|^2 > 0. \quad (29)$$

Hence we may write

$$\begin{aligned} \eta_1 &= \psi_1 \otimes (c\phi_2 + d\phi_3) \\ \eta_2 &= \psi_2 \otimes \phi_2 \\ \eta_3 &= b\psi_2 \otimes \phi_3. \end{aligned} \quad (30)$$

With these results we see for example that a completely factorizable orthogonal basis, i.e. a type (0, 4) basis, for \mathcal{H} is given by

$$\mathcal{B}_{0,4} = \{\psi_1 \otimes \phi_2, \psi_1 \otimes \phi_3, \psi_2 \otimes \phi_2, \psi_2 \otimes \phi_3\}. \quad (31)$$

Now consider a non-zero vector η_4 . This may be written in the form

$$\eta_4 = \alpha\psi_1 \otimes \phi_2 + \beta\psi_1 \otimes \phi_3 + \gamma\psi_2 \otimes \phi_2 + \delta\psi_2 \otimes \phi_3 \quad (32)$$

with

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 > 0. \quad (33)$$

If the η_i form an orthogonal, type (1, 3) basis then we must have

$$(\eta_1, \eta_4) = (\eta_2, \eta_4) = (\eta_3, \eta_4) = 0 \quad (34)$$

plus the micro-singularity condition discussed above that η_4 is entangled relative to the primordial basis, i.e.

$$C_4 \equiv \alpha\delta - \beta\gamma \neq 0. \quad (35)$$

The orthogonality conditions give

$$\begin{aligned} (\eta_2, \eta_4) = 0 &\Rightarrow \gamma = 0 \\ (\eta_3, \eta_4) = 0 &\Rightarrow \delta = 0. \end{aligned} \quad (36)$$

This gives

$$C_4 = 0 \quad (37)$$

however, which is inconsistent with (35), and hence the theorem is proved. \square

The above discussion of bases has some implications concerning operators. If \hat{A} is any Hermitian operator on \mathcal{H} with non-degenerate eigenvalues then there is a unique orthonormal

basis \mathcal{B}_A for \mathcal{H} formed from the normalized eigenvectors ψ_α of \hat{A} [11]. Suppose that \mathcal{H} is factorizable with bi-partite factorization

$$\mathcal{H} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)}, d_1, d_2 > 1 \quad (38)$$

Then \mathcal{B}_A will be of type (r, s) relative to this factorization, where $r + s = d_1 d_2$, for some non-negative integers r and s . For any orthonormal basis for \mathcal{H} , such a classification is unique, and therefore we conclude that any non-degenerate Hermitian operator on \mathcal{H} can be assigned a unique classification (r, s) relative to a given factorization of \mathcal{H} .

A conclusion from our results above is that there are no type $(1, 3)$ Hermitian operators acting on elements of a two-qubit system.

If a Hermitian operator \hat{A} has two or more degenerate eigenvalues then there is no uniqueness in the construction of an orthonormal basis from its eigenvectors. In such a case, more Hermitian operators \hat{B}, \hat{C}, \dots , commuting with \hat{A} and with each other need to be found in order to form a complete commuting set [11] $\mathcal{S} \equiv \{\hat{A}, \hat{B}, \hat{C}, \dots\}$. A complete commuting set then gives a unique orthonormal basis \mathcal{B}_S for \mathcal{H} , which will be of unique type (r, s) relative to the factorization (38). Hence we can classify \mathcal{S} as being of type (r, s) relative to the factorization (38) of the Hilbert space. This will be an important classification in physical situations involving physically identifiable factor spaces, such as qubit registers in quantum computers.

7. Concluding remarks

It has been observed by various researchers that even low-dimensional systems such as the two-qubit system still give surprises. We have found this to be the case in our investigation of factorizability of bases. Preliminary investigations have suggested that in more complicated systems, such as a bi-partite system of the form $\mathcal{H} \equiv \mathcal{H}_1^{(3)} \otimes \mathcal{H}_2^{(2)}$, the investigation of factorizability of bases becomes harder rapidly because of an increased number of combinatorial possibilities. It is not clear at this stage for example whether the non-existence of any type $(1, 3)$ basis or operator in the two-qubit case has analogues in higher dimensional systems. We have not to date found any type $(1, 5)$ basis for $\mathcal{H} \equiv \mathcal{H}_1^{(3)} \otimes \mathcal{H}_2^{(2)}$. A proof that one does not exist has not been found yet, although it appears intuitively obvious. Essentially, the single entangled basis element η in such a basis type would be of the form $\eta = \psi_1 \otimes \phi_1 + \psi_2 \otimes \phi_2 + \dots$, in obvious notation, such that $(\psi_1, \psi_2)_1 = 0$, etc, but because η has to be orthogonal to the subspace spanned by the five mutually orthogonal and separable basis vectors $\eta_1, \eta_2, \dots, \eta_5$, there is simply no 'space' for such an η to exist. From another point of view, it can be seen that although any single vector in a Hilbert space defines a one-dimensional subspace, even when it is entangled, the property of entanglement itself requires at least two dimensions for its operational definition. This argument leads us to conjecture that in general, there is no type $(1, d_1 d_2 - 1)$ basis for $\mathcal{H} \equiv \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)}$.

Acknowledgments

JE acknowledges an EPSRC research studentship. We are grateful to A Sudbery for comments.

References

- [1] Podolsky B, Einstein A and Rosen N 1935 Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **47** 777–80
- [2] Bohr N 1935 Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **48** 696–702

-
- [3] Eakins J and Jaroszkiewicz G 2002 The quantum universe *Preprint* quant-ph/0203020
 - [4] Markopoulou F and Smolin L 1997 Causal evolution of spin networks *Nucl. Phys. B* **508** 409–30
 - [5] Markopoulou F 2000 Quantum causal histories *Class. Quantum Grav.* **17** 2059–72
 - [6] Zizzi P A 2001 *The early universe as a quantum growing network* *Preprint* gr-qc/0103002
 - [7] Fink H and Leschke H 2000 Is the universe a quantum object? *Found. Phys. Lett.* **13** 345–56
 - [8] Zizzi P A 2000 Holography, quantum geometry, and quantum information theory *Entropy* **2** 39–69
 - [9] Fei S-M, Alberverio S and Goswami D 2001 Separability of rank two quantum states *Phys. Lett. A* **286** 91–6
 - [10] Jaroszkiewicz G 2001 The running of the universe and the quantum structure of time *Preprint* quant-ph/0105013
 - [11] Peres A 1993 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer)